THE FUNCTIONAL EQUATION $f[f(x)] = g(x)$.

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The object of this paper is to establish certain existence theorems concerning the solution of the functional equation

$$(A) \qquad f[f(x)] = g(x),$$

where

$$g(x) \equiv a_1x + a_2x^2 + \cdots$$

is a given analytic function defined in the neighborhood of the origin and vanishing there and such that $| a_1 | = 1$ and $a_1^n \neq 1$ for any positive integral value of n. It is easily seen that under these restrictions on $g(x)$ the equation (A) has two and only two formal solutions. It is here shown that functions $g(x)$ exist such that both of the solutions of (A) are divergent or one is divergent and the other convergent or both are convergent.

The method of proof used in this paper is the method used by the author in a paper† dealing with Schroeder's functional equation

$$\phi[f(x)] = a_1\phi(x),$$

where the given function

$$f(x) \equiv a_1x + a_2x^2 + \cdots, \quad \text{where} \quad | a_1 | = 1 \quad \text{and} \quad a_1^n \neq 1$$

for any positive integral value of n, is analytic about the origin. The latter equation and the equation considered in the present paper are closely connected. In particular, if the equation (A) has one divergent solution, then every formal solution of the equation

$$\phi[g(x)] = a_1\phi(x)$$

is divergent. For, if the latter had one convergent solution, $\phi_1(x)$, then

$$f_1(x) \equiv \phi_1^{-1}[c_1 \cdot \phi_1(x)] \quad \text{and} \quad f_2(x) \equiv \phi_1^{-1}[d_1 \cdot \phi_1(x)],$$

where $c_1 = \sqrt{a_1}$, $d_1 = -\sqrt{a_1}$ and $\phi_1^{-1}(x)$ denotes the inverse function of $\phi_1(x)$, would be both convergent solutions of (A) since (symbolically)‡

$$\phi_1^{-1}c_1\phi_1 \cdot \phi_1^{-1}c_1\phi_1 = \phi_1^{-1}d_1\phi_1 \cdot \phi_1^{-1}d_1\phi_1 = \phi_1^{-1}a_1\phi_1 = g.$$

* Read in part before the American Mathematical Society, October 30, 1915.

† See Transactions of the Am. Math. Soc., vol. 18 (1917), p. 185.

‡ The symbol fg denotes the function $f[g(x)]$.

This contradicts the hypothesis that the equation (A) has one divergent solution. Thus, Theorem 1 or Theorem 2 of the present paper implies the first theorem of the paper mentioned above which states the existence of divergent formal solutions of the Schroeder functional equation with

$$|a_1| = 1, \quad a_1^n \neq 1, \quad n = 1, 2, 3, \cdots,$$

and a suitably given function. However, it is easily shown that the latter theorem implies neither Theorem 1 nor Theorem 2 below. To show this let

$$f_1(x) \equiv a_1x + a_2x^2 + \cdots, \quad |a_1| = 1, \quad a_1^n \neq 1, \quad n = 1, 2, 3, \cdots,$$

be an analytic function defined about the origin and such that the functional equation

$$\phi[f_1(x)] = a_1\phi(x)$$

has no analytic solution $\phi(x)$. Let

$$g_1(x) \equiv f_1[f_1(x)] \equiv b_1x + b_2x^2 + \cdots.$$

Then $g_1(x)$ is analytic about the origin and, since $b_1 = a_1^2$, $|b_1| = 1$ and $b_1^n \neq 1$ for any positive integral value of n. Further, the Schroeder functional equation

$$\phi[g_1(x)] = b_1\phi(x)$$

has no analytic solution. For, if $\phi_1(x)$ were such a solution then (symbolically)

$$\phi_1f_1f_1 = b_1\phi_1 \quad \text{or} \quad \phi_1f_1\phi_1^{-1}\phi_1f_1\phi_1^{-1} = a_1a_1,$$

where again $\phi_1^{-1}(x)$ denotes the inverse of $\phi(x)$. From the last equation it follows by equating coefficients of like powers of x that

$$\phi_1f_1\phi_1^{-1} = a_1,$$

that is, $\phi_1(x)$ is a convergent solution of the equation

$$\phi[f_1(x)] = a_1\phi(x),$$

which is in contradiction to the definition of $f_1(x)$. Thus, although the Schroeder functional equation

$$\phi[g_1(x)] = b_1\phi(x), \quad \text{where} \quad |b_1| = 1 \quad \text{and} \quad b_1^n \neq 1$$

for any positive integral value of n, has no convergent solution, the functional equation

$$f[f(x)] = g_1(x)$$

has a convergent solution, namely $f_1(x)$.

In the case that the given function $g(x) \equiv x$, the functional equation in question reduces to the equation

$$f[f(x)] = x,$$

and it is well known that the latter equation has an infinite number of divergent solutions and an infinite number of convergent solutions. The latter equation is a special case of Babbage's functional equation

$$f^n(x) = x, \text{ where } f^n(x) \equiv f[f^{n-1}(x)].*$$

THEOREM. *There exists an analytic function* $g(x) \equiv a_1x + a_2x^2 + \cdots$, *defined in the neighborhood of the origin,* $|a_1| = 1$, $a_1^n \neq 1$, $n = 1, 2, 3, \cdots$, *such that the functional equation* $f[f(x)] = g(x)$ *has no solution which is analytic about the origin, i. e., every formal solution,* $f(x) \equiv c_1x + c_2x^2 + \cdots$, *is divergent for all values of* x *different from zero.*

Proof: Let

$$g_1(x) \equiv \alpha_1 x + \alpha_2 x^2 + \cdots, \qquad |\alpha_1| = 1, \quad \alpha_1^n \neq 1, \qquad n = 1, 2, \cdots,$$

be any function of x which is analytic about $x = 0$, and let

$$\phi(x) = \gamma_1 x + \gamma_2 x^2 + \cdots$$

be any formal solution of the equation

$$\phi[\phi(x)] = g_1(x).$$

Then we have

$$\gamma_1^2 = \alpha_1, \qquad \gamma_2 = \frac{\alpha_2}{\gamma_1(1+\gamma_1)},$$

$$\gamma_3 = \frac{\gamma_1^2(1+\gamma_1)^2\alpha_3 - 2\gamma_1\alpha_2^2}{\gamma_1^3(1+\gamma_1)^2(1+\gamma_1^2)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\gamma_{n+1} = \frac{\gamma_1^i(1+\gamma_1)^j\cdots(1+\gamma_1^{n-1})\alpha_{n+1} + P_{n+1}(\gamma_1, \alpha_2, \cdots, \alpha_n)}{\gamma_1^{i+1}(1+\gamma_1)^j\cdots(1+\gamma_1^{n-1})(1+\gamma_1^n)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where $P_{n+1}(\gamma_1, \alpha_2, \cdots, \alpha_n)$ is a polynomial in γ_1, α_i, $i = 2, 3, \cdots, n$, and where $i, j, \cdots$ are integers.

We proceed to prove the theorem by determining a set of values $[a_i]$

* Cf. the following: A. A. Bennett, Annals of Math., vol. 17 (1915), p. 37; G. A. Pfeiffer, Amer. Jour. of Math., vol. 17 (1915), p. 421; J. F. Ritt, Annals of Math., vol. 17 (1916), p. 113. (This article is, however, not concerned with analytic solutions.); L. Leau, S. M. F. Bull., vol. 26 (1898), p. 5; E. M. Lemeray, S. M. F. Bull., vol. 26 (1898), p. 10.

As related to the present subject the reader is referred to the following papers by the writer: "On the Conformal Mapping of Curvilinear Angles. The Functional Equation $\phi[f(x)] = a_1\phi(x)$," Trans. Amer. Math. Soc., vol. 18 (1917); "On the Conformal Geometry of Analytic Arcs," Amer. Journ. of Math., vol. 17 (1915); also, to the paper by A. A. Bennett entitled "The Iteration of Functions of One Variable," Annals of Math., 2d series, vol. 17 (1915), and to the literature listed in the papers just mentioned.

for the α_i such that the a_i are the coefficients of a convergent power series, and $| a_1 | = 1$ and $a_1^n \neq 1$, $n = 1, 2, \cdots$, and such that c_i, d_i $(i = 2, 3, \cdots)$, the corresponding values of γ_i when $\gamma_1 = +\sqrt{a_1}$ and $-\sqrt{a_1}$ respectively, are the coefficients of two power series each with a zero radius of convergence.

Let $F_{n+1}(\gamma_1, \alpha_2, \cdots, \alpha_{n+1})$ denote the rational integral function

$$\gamma_1^i(1 + \gamma_1)^j(1 + \gamma_1^2)^k \cdots (1 + \gamma_1^{n-1})\alpha_{n+1} + P_{n+1}(\gamma_1, \alpha_2, \cdots, \alpha_n).$$

Let $a^{(1)}$ be a primitive m-th root of -1, where m is even; in particular, let

$$a^{(1)} = \cos\frac{l}{m}\pi + i\sin\frac{l}{m}\pi,$$

where l is an odd positive integer $< m$ and where m is a positive integral power of 2. Then

$$b^{(1)} = \cos\frac{l + m}{m}\pi + i\sin\frac{l + m}{m}\pi$$

is also a primitive m-th root of -1 and hence the coefficients of α_{m+1} in $F_{m+1}(a^{(1)}, \alpha_2, \cdots, \alpha_{m+1})$ and $F_{m+1}(b^{(1)}, \alpha_2, \cdots, \alpha_{m+1})$ do not vanish. Therefore there exist definite values of $\alpha_2, \cdots, \alpha_{m+1}$, say $a_2, \cdots, a_{m+1}$, such that

$$| a_i - a_i' | < \delta, \; i = 2, 3, \cdots, m + 1,$$

where δ is an arbitrary positive number and the a_i', $i = 2, 3, \cdots$, are the coefficients of any convergent power series, and such that

$$F_{m+1}(t, a_2, \cdots, a_{m+1}) \neq 0$$

for both $| t - a^{(1)} | < \epsilon_1$ and $| t - b^{(1)} | < \epsilon_1$, where ϵ_1 is a positive number sufficiently small. In particular, $a_2, \cdots, a_m, a_i'$ $(i = 2, 3, \cdots)$ may all be taken equal to zero.

Let $\epsilon_1' \leqq \epsilon_1$ be a positive number such that no root of ± 1 of order $< m$ is in either of the ranges $| t - a^{(1)} | \leqq \epsilon_1'$, $| t - b^{(1)} | \leqq \epsilon_1'$. Such a number ϵ_1' obviously exists since there is only a finite number of such roots of ± 1. Since

$$\left| \frac{F_{m+1}(t, a_2, \cdots, a_{m+1})}{t^{i+1}(1 + t)^j(1 + t^2)^k \cdots (1 + t^{m-1})} \right|$$

has a lower bound $\mu_{m+1} > 0$ for both $| t - a^{(1)} | \leqq \epsilon_1'$ and $| t - b^{(1)} | \leqq \epsilon_1'$, there exists a positive number $\epsilon_1'' \leqq \epsilon_1'$, such that

$$\left| \frac{F_{m+1}(t, a_2, \cdots, a_{m+1})}{t^{i+1}(1 + t)^j(1 + t^2)^k \cdots (1 + t^{m-1})(1 + t^m)} \right| > \lambda_{m+1},$$

where λ_{m+1} is an arbitrary positive number, for $0 < | t - a^{(1)} | < \epsilon_1''$ and $0 < | t - b^{(1)} | < \epsilon_1''$, $| t | = 1$, and no root of ± 1 of order $< m$ is in either of the ranges $| t - a^{(1)} | < \epsilon_1''$, $| t - b^{(1)} | < \epsilon_1''$.

Let $p > m$ be a positive integral power of 2 and let $a^{(2)}$ be a primitive p-th root of -1 such that $| a^{(2)} - a^{(1)} | < \frac{\epsilon_1''}{2}$. Such a number $a^{(2)}$ is easily determined as follows: We have

$$a^{(1)} = \cos \frac{l}{m} \pi + i \sin \frac{l}{m} \pi,$$

l = an odd positive integer $< m$ and m = a positive integral power of 2. Let p be a positive integral power of 2 and $> m$ and $\frac{2\pi}{\epsilon''} \left(\frac{\epsilon''}{2} \text{ is here assumed to be less than unity} \right)$ and let k be the odd positive integer which is such that

$$\frac{lp}{m} - 1 < k \leqq \frac{lp}{m} + 1.$$

Then it is easily shown that the number $\cos \frac{k}{p} \pi + i \sin \frac{k}{p} \pi$ may be taken as the number $a^{(2)}$. Also, there exists a primitive p-th root of -1, say $b^{(2)}$, such that $| b^{(2)} - b^{(1)} | < \frac{\epsilon_1''}{2}$. In particular,

$$b^{(2)} = \cos \frac{k + p}{p} \pi + i \sin \frac{k + p}{p} \pi$$

is such a number.

Now proceeding as above, there exists a positive number $\epsilon_2 < \frac{\epsilon_1''}{2}$ such that $F_{p+1}(t, a_2, \cdots, a_{p+1}) \neq 0$ for both $| t - a^{(2)} | < \epsilon_2$ and $| t - b^{(2)} | < \epsilon_2$, where the a_i, $i = 2, \cdots, m + 1$ are those fixed upon above and $| a_i - a_i' | < \delta$, $i = 2, \cdots, p + 1$. Again, the a_i, $i = m + 2, m + 3, \cdots, p$, may be all taken equal to zero. Then let $\epsilon_2' \leqq \epsilon_2$ be a positive number such that no root of ± 1 of order $< p$ is in either of the ranges

$$| t - a^{(2)} | \leqq \epsilon_2', \qquad | t - b^{(2)} | \leqq \epsilon_2'.$$

Then we have, as above,

$$\left| \frac{F_{p+1}(t, a_2, \cdots, a_{p+1})}{t^{e+1}(1 + t)^f(1 + t^2)^g \cdots (1 + t^{p-1})(1 + t^p)} \right| > \lambda_{p+1},$$

where λ_{p+1} is an arbitrary positive number, for

$$0 < | t - a^{(2)} | < \epsilon_2'' \quad \text{and} \quad 0 < | t - b^{(2)} | < \epsilon_2'', \qquad | t | = 1,$$

where ϵ_2'' is a positive number $\leqq \epsilon_2'$, and no root of ± 1 of order $< p$ is in either of the ranges

$$| t - a^{(2)} | < \epsilon_2'', \qquad | t - b^{(2)} | < \epsilon_2''.$$

Again, let r be an even integer greater than p and let $a^{(3)}$ and $b^{(3)}$ be two primitive r-th roots of -1 such that

$$| a^{(3)} - a^{(2)} | < \frac{\epsilon_2''}{2} \qquad \text{and} \qquad | b^{(3)} - b^{(2)} | < \frac{\epsilon_2''}{2}$$

and continue as above. Thus corresponding to the terms of the infinite sequence $m, p, r, \cdots$, where $m, p, r, \cdots$ are positive integers such that $m < p < r < \cdots$, we have the inequalities

$$\left| \frac{F_{m+1}(t, a_2, \cdots, a_{m+1})}{t^{i+1}(1+t)^j(1+t^2)^k \cdots (1+t^{m-1})(1+t^m)} \right| > \lambda_{m+1}$$

for both $0 < | t - a^{(1)} | < \epsilon_1''$ and $0 < | t - b^{(1)} | < \epsilon_1''$,

$$\left| \frac{F_{p+1}(t, a_2, \cdots, a_{p+1})}{t^{e+1}(1+t)^f(1+t^2)^g \cdots (1+t^{p-1})(1+t^p)} \right| > \lambda_{p+1}$$

for both $0 < | t - a^{(2)} | < \epsilon_2''$ and $0 < | t - b^{(2)} | < \epsilon_2''$,

$$\left| \frac{F_{r+1}(t, a_2, \cdots, a_{r+1})}{t^{u+1}(1+t)^v(1+t^2)^w \cdots (1+t^{r-1})(1+t^r)} \right| > \lambda_{r+1}$$

for both $0 < | t - a^{(3)} | < \epsilon_3''$ and $0 < | t - b^{(3)} | < \epsilon_3''$,

. . . .,

where the λ_i are arbitrary positive numbers.

Now the range

$$| t - a^{(i)} | < \epsilon_i'', \qquad | t | = 1,$$

is contained in the range

$$| t - a^{(i-1)} | < \epsilon_{i-1}'', \qquad | t | = 1,$$

where $\epsilon_i'' < \dfrac{\epsilon_{i-1}''}{2}$. Consequently, there is just one number common to all of the ranges $| t - a^{(i)} | < \epsilon_i''$, $| t | = 1$. No root of ± 1 can be common to all of these ranges, since any root of ± 1 which is contained in a certain range is not contained in any of the succeeding ones. Let a be the number common to all of these ranges. Then a and $a_1 = a^2$ each has unity for its modulus, and $a^n \neq \pm 1$ and $a_1{}^n \neq 1$ for $n = 1, 2, 3, \cdots$. Likewise, the ranges $| t - b^{(i)} | < \epsilon_i''$, $|t| = 1$, have just one number in common, say b. Evidently

$$b^2 = a^2 = a_1 \qquad \text{and} \qquad b^n \neq \pm 1, \qquad n = 1, 2, 3, \cdots.$$

Let the positive numbers

$$\lambda_{m+1},\ \lambda_{p+1},\ \lambda_{r+1},\ \cdots$$

be taken so that the sequence

$$\sqrt[1]{\lambda_{m+1}},\ \sqrt[2]{\lambda_{p+1}},\ \sqrt[3]{\lambda_{r+1}},\ \cdots$$

is not bounded, then the sequences

$$\sqrt[1]{|c_{m+1}|},\ \sqrt[2]{|c_{p+1}|},\ \sqrt[3]{|c_{r+1}|},\ \cdots;\quad \sqrt[1]{|d_{m+1}|},\ \sqrt[2]{|d_{p+1}|},\ \sqrt[3]{|d_{r+1}|},\ \cdots$$

are not bounded, and, consequently, the series

$$\sum_1^\infty c_i x^i, \qquad \sum_1^\infty d_i x^i$$

are divergent for all values of $x \neq 0$. The function $g(x)$ of the theorem is $a_1x + a_2x^2 + a_3x^3 + \cdots$. Q. E. D.

For the function $g(x)$ just determined the two formal solutions of the given functional equation are divergent. For some functions $g(x)$ there exist one convergent solution and one divergent solution. For others both solutions are convergent in the neighborhood of the origin. We proceed to prove the

THEOREM. *There exists an analytic function $g(x) \equiv a_1x + a_2x^2 + \cdots$, defined about the origin, $|a_1| = 1$, $a_1^n \neq 1$, $n = 1, 2, 3, \cdots$, such that the functional equation $f[f(x)] = g(x)$ has one and only one solution which is analytic about the origin.*

Proof: Consider the two expressions

$$H \equiv \gamma_1(\gamma_1x + \gamma_2x^2 + \cdots + \gamma_nx^n) + \gamma_2(\gamma_1x + \gamma_2x^2 + \cdots + \gamma_nx^n)^2 + \cdots + \gamma_n(\gamma_1x + \gamma_2x^2 + \cdots + \gamma_nx^n)^n,$$

$$J \equiv -\gamma_1(-\gamma_1x + \gamma_2'x^2 + \cdots + \gamma_n'x^n) + \gamma_2'(-\gamma_1x + \gamma_2'x^2 + \cdots + \gamma_n'x^n)^2 + \cdots + \gamma_n'(-\gamma_1x + \gamma_2'x^2 + \cdots + \gamma_n'x^n)^n,$$

and then consider the set of equations obtained by equating the coefficients of like powers of x in H and J. This set of equations is as follows:

$$\gamma_1\gamma_2 + \gamma_2\gamma_1^2 = -\gamma_1\gamma_2' + \gamma_2'\gamma_1^2,$$

$$\gamma_1\gamma_3 + 2\gamma_1\gamma_2^2 + \gamma_3\gamma_1^3 = -\gamma_1\gamma_3' - 2\gamma_1\gamma_2'^2 - \gamma_3'\gamma_1^3,$$

$$\gamma_1\gamma_4 + 2\gamma_1\gamma_2\gamma_3 + \gamma_2^3 + 3\gamma_1\gamma_3^2 + \gamma_4\gamma_1^4 = -\gamma_1\gamma_4' - 2\gamma_1\gamma_2'\gamma_3' + \gamma_2'^3 - 3\gamma_1\gamma_3'^2 + \gamma_4'\gamma_1^4,$$

. . . .

$$\gamma_1\gamma_{2n} + P_{2n}(\gamma_1, \gamma_2, \cdots, \gamma_{2n-1}) + \gamma_{2n}\gamma_1^{2n}$$
$$= -\gamma_1\gamma'_{2n} + P'_{2n}(\gamma_1, \gamma_2', \cdots, \gamma'_{2n-1}) + \gamma'_{2n}\gamma_1^{2n},$$
$$\gamma_1\gamma_{2n+1} + P_{2n+1}(\gamma_1, \gamma_2, \cdots, \gamma_{2n}) + \gamma_{2n+1}\gamma_1^{2n+1}$$
$$= -\gamma_1\gamma'_{2n+1} + P'_{2n+1}(\gamma_1, \gamma_2', \cdots, \gamma_{2n}) - \gamma'_{2n+1}\gamma_1^{2n+1},$$
$$\cdots\cdots\cdots\cdots\cdots,$$

where P_i, P_i' are polynomials in the indicated arguments.

Solving this system of equations, we determine γ'_{2n} and γ'_{2n+1} as functions of $\gamma_1, \gamma_2, \cdots, \gamma_{2n-1}$ and $\gamma_1, \gamma_2, \cdots, \gamma_{2n}$ respectively. We have

$$\nu_2' = \frac{\gamma_1(1+\gamma_1)\gamma_2}{-\gamma_1(1-\gamma_1)},$$

$$\gamma_3' = \frac{\gamma_1^3(1-\gamma_1)^2(1+\gamma_1^2)\gamma_3 + \boldsymbol{P}_3(\gamma_1, \gamma_2)}{-\gamma_1^3(1-\gamma_1)^2(1+\gamma_1^2)},$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$\gamma'_{2n} = \frac{\gamma_1^i(1-\gamma_1)^j(1+\gamma_1^2)^k(1-\gamma_1^3)^l\cdots(1+\gamma_1^{2n-2})(1+\gamma_1^{2n-1})\gamma_{2n} + \boldsymbol{P}_{2n}(\gamma_1, \gamma_2, \cdots, \gamma_{2n-1})}{-\gamma_1^i(1-\gamma)^j(1+\gamma_1^2)^k(1-\gamma_1^3)^l\cdots(1+\gamma_1^{2n-2})(1-\gamma_1^{2n-1})},$$

$$\gamma'_{2n+1} = \frac{\gamma_1^q(1-\gamma_1)^r(1+\gamma_1^2)^s(1-\gamma_1^3)^t\cdots(1-\gamma_1^{2n-1})(1+\gamma_1^{2n})\gamma_{2n+1} + \boldsymbol{P}_{2n+1}(\gamma_1, \gamma_2, \cdots, \gamma_{2n})}{-\gamma_1^q(1-\gamma_1)^r(1+\gamma_1^2)^s(1-\gamma_1^3)^t\cdots(1-\gamma_1^{2n-1})(1+\gamma_1^{2n})},$$

$$\cdots\cdots\cdots\cdots\cdots,$$

where $\boldsymbol{P}_i$ are polynomials in the indicated arguments.

The preceding method of proof will now be used to show that values of $\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots$ can be found, such that $\Sigma\gamma_i x^i$ is a convergent power series for those values of γ_i, and such that the corresponding values of $\gamma_2', \gamma_4', \cdots, \gamma'_{2n}, \cdots$ as determined by the above set of equations are the coefficients of a power series with a zero radius of convergence. Thus, we consider the set of equations which determine γ'_{2n}. We write

$$F_{2n} \equiv \gamma_1^i(1-\gamma_1)^j(1+\gamma_1^2)^k(1-\gamma_1^3)^l\cdots(1-\gamma_1^{2n-2})(1+\gamma_1^{2n-1})\gamma_{2n} + \boldsymbol{P}_{2n}(\gamma_1, \gamma_2, \cdots, \gamma_{2n-1}).$$

In this proof primitive roots of $+1$ of odd order are used. If $c^{(1)}$ is a primitive m-th root of $+1$, m = an odd positive integer, then $(c^{(1)})^s \neq \pm 1$ for s a positive integer $< m$, and therefore the coefficient of γ_{m+1} in $F_{m+1} \neq 0$ ($m+1$, even) for $\gamma_1 = c^{(1)}$. Following the same line of argument as above, we obtain *one* set of ranges $0 < |t - c^{(1)}| < \epsilon_1''$, $0 < |t - c^{(2)}| < \epsilon_2''$, $0 < |t - c^{(3)}| < \epsilon_3''$, $\cdots$; $|t| = 1$, where $c^{(i)}$, $i = 1, 2, 3, \cdots$, are primitive roots of $+1$ of odd orders, $m, p, r, \cdots$,

respectively, and such that the order of $c^{(k)}$ is greater than the order of $c^{(j)}$ if $k > j$, which close down on one number c_1. Further, these ranges are such that $|\gamma_{2j}'| > \lambda_i$ for $0 < |t - c^{(i)}| < \epsilon_i''$, where λ_i is an arbitrary positive number and $c^{(i)}$ is a certain primitive root of $+1$ of order $2j - 1$ chosen in a manner exactly analogous to the method of choice used in the preceding proof. For definiteness we take the numbers $m, p, r, \cdots$ in this case (the notation is that used in the preceding proof) to be prime numbers > 2 and such that $m < p < r < \cdots$, and let

$$c^{(1)} = \cos\frac{l}{m}\pi + i\sin\frac{l}{m}\pi,$$

where l is an even positive integer $< m$, a prime number > 2. Then

$$c^{(2)} = \cos\frac{k}{p}\pi + i\sin\frac{k}{p}\pi,$$

where p is a prime number $> m$ and $\frac{2\pi}{\epsilon''}$ and where k is the even integer which is such that

$$\frac{lp}{m} - 1 < k \leqq \frac{lp}{m} + 1.$$

Then, again, it is readily shown that $|c^{(2)} - c^{(1)}| < \frac{\epsilon_1''}{2}$. We proceed similarly in the choice of $c^{(3)}$, a primitive r-th root of unity. The notation used here indicates that γ_{2j}' is of order index $2j$ in the sequence $[\gamma_n']$ and of order index i in the particular sub-sequence of the sequence $[\gamma_n']$ used in establishing the inequalities $|\gamma_{2j}'| > \lambda_i$. Then, as above, by properly taking the λ_i the set of values $[c_{2j}]$ of γ_{2j} is determined such that the corresponding values of γ_{2j}', c_{2j}' say, thus determined are the coefficients of a divergent power series while $\Sigma c_n x^n$ is a convergent power series, the c_n not among the chosen c_{2j} being arbitrary, except that they be the coefficients of a convergent power series; in particular, they may all be taken equal to zero.

Let

$$f_1(x) \equiv \Sigma c_n x^n,$$

then

$$g(x) \equiv f_1[f_1(x)]$$

is analytic about the origin. The other formal solution of the equation

$$f[f(x)] = g(x) \equiv f_1[f_1(x)]$$

is $\Sigma c_n' x^n$, where the c_n' are the values of γ_n' determined by the above set of equalities in γ'_n and γ_n when γ_n are put equal respectively to the cor-

responding particular values c_n just fixed upon. $\Sigma c_n' x^n$ is divergent for all values of $x \neq 0$, and

$$g(x) \equiv f_1[f_1(x)]$$

is a function as required by the theorem. Q. E. D.

To the two theorems just proved we add the following easily proved theorem as a natural completion of the above:

THEOREM. *There exists a function*

$$g(x) \equiv a_1 x + a_2 x^2 + \cdots, \qquad |a_1| = 1, \qquad a_1^n \neq 1, \qquad n = 1, 2, 3, \cdots,$$

which is analytic about the origin and which is such that the functional equation $f[f(x)] = g(x)$ *has two solutions which are analytic about the origin.*

Proof: The simplest example of a $g(x)$ which proves this theorem is the linear function $a_1 x$ (a_1 arbitrary, except that the conditions of theorem are satisfied); the two solutions are $\sqrt{a_1}x$ and $-\sqrt{a_1}x$. Non-linear examples of a $g(x)$ are easily gotten by taking the transform of the function $a_1 x$ by any analytic function,

$$b(x) \equiv b_1 x + b_2 x^2 + \cdots, \; b_1 \neq 0,$$

defined about the origin; i. e.,

$$g(x) \equiv b[a_1(b^{-1}(x))],$$

where $b^{-1}(x)$ denotes the inverse of $b(x)$, is such a function. In this case the two solutions are $b[c_1(b^{-1}(x))]$ and $b[d_1(b^{-1}(x))]$, where

$$c_1 = \sqrt{a_1}, \qquad d_1 = -\sqrt{a_1}$$

and both are analytic in the neighborhood of the origin. That either solution satisfies the given functional equation is evident. For, we have, symbolically,

$$bc_1b^{-1}bc_1b^{-1} = bd_1b^{-1}bd_1b^{-1} = ba_1b^{-1}.$$

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